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An algebraic analysis of the connectivity of DeBruijn and shuffle-exchange digraphs

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Abstract

We study connectivity properties of d-ary deBruijn and shuffle-exchange digraphs by appealing to their algebraic structure. Our first result proves that both these families of digraphs are $(d-1)$-connected. The proof also leads to two substantially stronger results. Namely, we prove that for the order-n, d-ary deBruijn digraph (resp. the order-n, d-ary shuffle-exchange digraph), any set of *shuffle cycles* of total length less than $n(d-1)$ can be removed and the digraph remains strongly connected. The second extension characterizes the pairs of vertices in the d-ary deBruijn digraphs (resp. the d-ary shuffle-exchange digraphs) which have d disjoint paths between them. The central idea in the paper rests upon a new application of the group-theoretic relationship between shuffle-oriented digraphs, butterfly-like digraphs and hypercubes.

1. Introduction

Given a group G and a generating set S for G, the *Cayley digraph of G with respect to S* is a digraph with vertex set G and arc set $\{(g, gs)\colon g \in G,\ s \in S\}$. Akers and Krishnamurthy [1] proposed Cayley digraphs as appropriate models of interconnection networks for parallel computers due to their symmetry, desirable graph-theoretic attributes and potential for providing a unified framework for approaching various problems. Subsequently, Annexstein et al. [2] showed that *right quotient digraphs* of Cayley digraphs shared many structural and algorithmic properties with their associated Cayley digraphs. In particular, they demonstrated that the d-ary deBruijn digraph is a right quotient digraph of the d-ary butterfly digraph and similarly, the d-ary shuffle-exchange digraph is a right quotient digraph of the d-ary cube-connected cycles digraph. The underlying algebraic relationships facilitated the discovery of an efficient embedding of the butterfly graphs into the deBruijn graphs. In this paper, we further analyze these relationships and present a technique whereby the connectivity

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of right quotient digraphs can be analyzed in terms of the connectivity of the associated Cayley digraphs and left quotient digraphs.

We pursue our investigation by focusing on deBruijn and shuffle-exchange digraphs and prove the following results:

1. The connectivity of the d-ary deBruijn (shuffle-exchange) digraphs equals $d - 1$.
2. Any set of *shuffle cycles* of total length less than $n(d - 1)$ can be removed from the order-n, d-ary deBruijn (shuffle-exchange) digraph and it remains strongly connected.
3. We characterize the pairs of vertices in the d-ary deBruijn (shuffle-exchange) digraphs which have d disjoint paths between them, i.e., we show that there are d disjoint paths from u to v if and only if both u and v are of degree d and neither u nor v is adjacent to a vertex with a self-loop.

All three results are constructive based on the existence of disjoint paths in d-ary hypercubes (see Section 5.1). The first result is new only in the case of the shuffle-exchange digraphs. The second and third results are new for both deBruijn and shuffle-exchange digraphs. All of the results revolve around a central novel technique for studying the connectivity of right quotient digraphs via the connectivity of their associated Cayley digraphs and left quotient digraphs.

The fact that deBruijn digraphs have connectivity equal to their minimum degree has been shown previously by several authors through combinatorial means [4, 5, 14, 15]. A simple proof arises from the fact that the deBruijn digraphs can be generated by repeated line digraph iterations starting from an appropriate complete graph. However, this approach cannot be applied to shuffle-exchange digraphs and it is not clear whether the other results we derive in this paper can be obtained easily using only combinatorial arguments.

Soneoka et al. [14] have studied connectivity and diameter properties of a large class of generalized deBruijn digraphs. They prove a result closely related to our last result in the case of the deBruijn digraphs. Namely, they show that a *generalized* deBruijn digraph for which all vertices with self-loops are connected in a cycle has connectivity equal to its degree. (Their result holds as long as the size of the vertex set is greater than d^3, where d is the size of the underlying alphabet – see [14].) The proof technique in [14] is based on properties of cutsets of generalized deBruijn digraphs and is non-constructive. Also, their results do not apply to the shuffle-exchange digraphs.

2. Basic definitions

2.1. Graph-theoretic definitions

Let $\mathscr{G} = (V, E)$ be a directed graph, with vertex set V and arc set E. If $v \in V$, the set of vertices *incident from* v is $\{x \in V\colon (v, x) \in E,\ x \neq v\}$. The cardinality of this set is called the *out-degree* of v. Similarly, the set of vertices *incident to* v is $\{x \in V\colon (x, v) \in E,\ x \neq v\}$, and its cardinality is called the *in-degree* of v. If the out-degree equals the in-degree of v we speak of the *degree* of v.

We define a *path* from $v_0 \in V$ to $v_k \in V$ to be a finite sequence of abutting arcs

$$(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k).$$

(For each $i \in \{0, 1, \ldots, k-1\}$, $(v_i, v_{i+1}) \in E$.) Note that this definition does not preclude a path from containing cycles. A digraph is *strongly connected* if there is a path between any pair of distinct vertices. The *connectivity* of $\mathscr{G}$, denoted $\kappa(\mathscr{G})$, is the minimum number of vertices that must be removed so that the remaining digraph is not strongly connected or is trivial. By Menger's theorem (cf. [3]), this definition is equivalent to the following: For any pair u, v of distinct vertices, there are $\kappa(\mathscr{G})$ vertex-disjoint paths from u to v. In general, if u and v are two vertices of $\mathscr{G}$, then the *(u,v)-connectivity* is the maximum number of vertex-disjoint paths from u to v in $\mathscr{G}$.

In this paper, we will make use of the following characterization of connectivity: $\mathscr{G}$ has connectivity $\kappa(\mathscr{G})$ if, for any pair of distinct vertices x, y, and any set A of $\kappa(\mathscr{G}) - 1$ vertices such that $x, y \notin A$, there is a path from x to y that *avoids* A (i.e., does not pass through any vertices in A).

Finally, define the *direct product* of two digraphs, $\mathscr{G} = (V_1, E_1)$ and $\mathscr{H} = (V_2, E_2)$, denoted by $\mathscr{G} \times \mathscr{H}$, as follows. The vertex set is the cartesian product $V_1 \times V_2$. There is an arc from (v_1, v_2) to (v'_1, v'_2) when $v_1 = v'_1$ and $(v_2, v'_2) \in E_2$, or $v_2 = v'_2$ and $(v_1, v'_1) \in E_1$.

2.2. DeBruijn, butterfly and hypercube digraphs

In this section we provide the standard definitions of the main families of digraphs germane to this paper (see Fig. 1 for examples of standard depictions of these digraphs). We will develop our technique in the context of deBruijn digraphs and defer the exposition regarding the shuffle-exchange digraphs to Section 7.

DeBruijn digraphs: The order-(d,n) *deBruijn digraph* ($d \geq 2$, $n \geq 1$), denoted $\mathscr{D}_{d,n}$, has a vertex set consisting of all n-tuples, $[b_0, b_1, \ldots, b_{n-1}]$, where each $b_i \in \{0, 1, \ldots, d-1\}$. There is an arc from $[b_0, b_1, \ldots, b_{n-1}]$ to $[b_1, b_2, \ldots, b_{n-1}, b']$, for all $b' \in \{0, 1, \ldots, d-1\}$. When $b' = b_0$ the arc is called a *shuffle arc*.

Butterfly digraphs: The order-(d,n) *butterfly digraph* ($d \geq 2$, $n \geq 1$), denoted $\mathscr{B}_{d,n}$, has a vertex set consisting of the set of all pairs $\langle l; B \rangle$, where $l \in \{0, 1, \ldots, n-1\}$ and B is an n-tuple $[b_0, b_1, \ldots, b_{n-1}]$ with each $b_i \in \{0, 1, \ldots, d-1\}$. There is an arc from $\langle l_1; B_1 \rangle$ to $\langle l_2; B_2 \rangle$ if $l_2 = l_1 + 1 \bmod n$ and either $B_1 = B_2$ or $B_1 = [b_0, \ldots, b_{l_1}, \ldots, b_{n-1}]$ and $B_2 = [b_0, \ldots, b', \ldots, b_{n-1}]$, for all $b' \in \{0, 1, \ldots, d-1\}$.

Hypercube digraphs: The order-(d,n) *hypercube digraph* ($d \geq 2, n \geq 1$), denoted $\mathscr{Q}_{d,n}$, has a vertex set consisting of all n-tuples $[b_0, b_1, \ldots, b_{n-1}]$ where each $b_i \in \{0, 1, \ldots, d-1\}$. There is a pair of mutually opposing arcs between any two n-tuples when they differ in exactly one position, i.e., $[b_0, \ldots, b_i, \ldots, b_{n-1}]$ is adjacent to $[b_0, \ldots, b'_i, \ldots, b_{n-1}]$, for each $i \in \{0, 1, \ldots, n-1\}$ and $b'_i \neq b_i \in \{0, 1, \ldots, d-1\}$.

Another useful way of defining hypercubes is in terms of direct products. Let $\mathscr{K}_d$ denote the complete symmetric digraph on d vertices.[1] Then, $\mathscr{Q}_{d,n}$ is the n-fold direct product of $\mathscr{K}_d$, denoted by $\mathscr{K}_d^n$.

[1] The *complete symmetric digraph* has a pair of mutually opposing arcs for every pair of vertices.

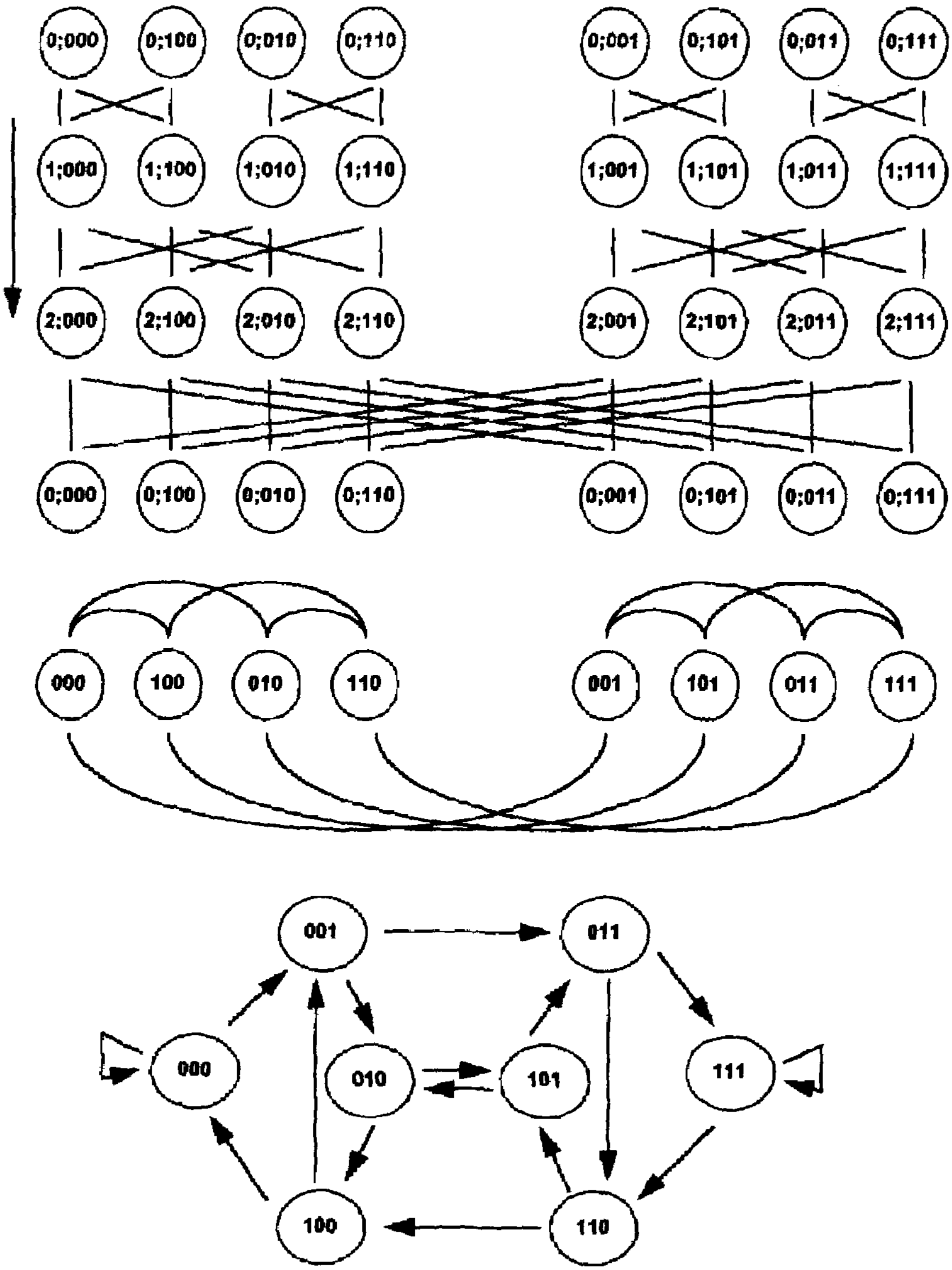

Fig. 1. Standard labelings of order-(2, 3) butterfly, hypercube and deBruijn digraphs.

2.3. Graphs defined by groups

For more information on basic group theory, we refer the reader to [7]. Given a group G and a generating set S for G (i.e., a subset of group elements with the property that every group element can be expressed as a product of the generators), the *Cayley digraph of G with respect to S*, denoted by $\Gamma(G, S)$, is a digraph with vertex set G and arc set $\{(g, gs)\colon g \in G,\ s \in S\}$.

Suppose H is a subgroup of G. A *right coset of H in G* is a set of the form Hg, $g \in G$. Similarly, a *left coset of H in G* is a set of the form gH, $g \in G$. We denote the set of all right or left cosets by G/H, appealing to context to disambiguate. The set of all right cosets (or left cosets) partitions G. In general, the partition by left cosets does not coincide with the partition by right cosets.

Two natural quotient digraphs can be formed from $\Gamma(G, S)$ given a subgroup H. (See Fig. 2 for a specific example illustrating their construction.) The *right quotient digraph*,[2] denoted by $\Gamma_{\mathrm{R}}(G/H, S)$, has a vertex set consisting of the *right* cosets of H in G, and arc set consisting of all ordered pairs of the form (Hg, Hgs), where $s \in S$.

The *left quotient digraph* (originally introduced in [13]), denoted by $\Gamma_{\mathrm{L}}(G/H, S)$, has a vertex set consisting of the *left* cosets of H in G. There is an arc from $g_1 H$ to $g_2 H$ $(g_1 H \neq g_2 H)$ if $g_1 h_1 s = g_2 h_2$ for some $h_1, h_2 \in H$ and $s \in S$. The following proposition is a consequence of the fact that multiplication on the left by any element of the group defines an automorphism of the Cayley digraph.

Proposition 1. *Let $\Gamma(G, S)$ be a Cayley digraph and suppose H is a subgroup of G. Then, the induced subgraphs on the left cosets of H in G are all isomorphic.*

3. Preliminary results

In this section, we recall some results from [2] which describe the algebraic structure of the butterfly digraphs and the deBruijn digraphs. Denote the additive group of integers modulo n by $Z_n = \{0, 1, \dots, n-1\}$. Define $G_{d,n} = Z_d \ wr\ Z_n$, the *wreath product*[3] of Z_d by Z_n, as follows. The elements of $G_{d,n}$ are pairs in which the first component of the pair is a single element from Z_n and the second component of the pair is an n-tuple of elements from Z_d. A typical element of $G_{d,n}$ is written as

$$\langle a; b_0, b_1, \dots, b_{n-1} \rangle,$$

where $a \in Z_n$ and $b_i \in Z_d$ $(0 \leqslant i \leqslant n-1)$. Multiplication of $g_1 = \langle a; b_0, b_1, \dots, b_{n-1} \rangle$ by $g_2 = \langle a'; b'_0, b'_1, \dots, b'_{n-1} \rangle$ produces an element $g_3 = g_1 g_2$ such that:

- The first component of g_3 is the sum of the first components of g_1 and g_2.
- The second component of g_3 is obtained by performing a' left-circular shifts on the second component of g_1 and then performing a component-wise addition of the result with the second component of g_2.

[2] In [2], these graphs were called *group-action graphs*. They are also commonly known as *Schreier Coset Graphs*.

[3] Our definition of the wreath product is commonly known as the *standard wreath product*.

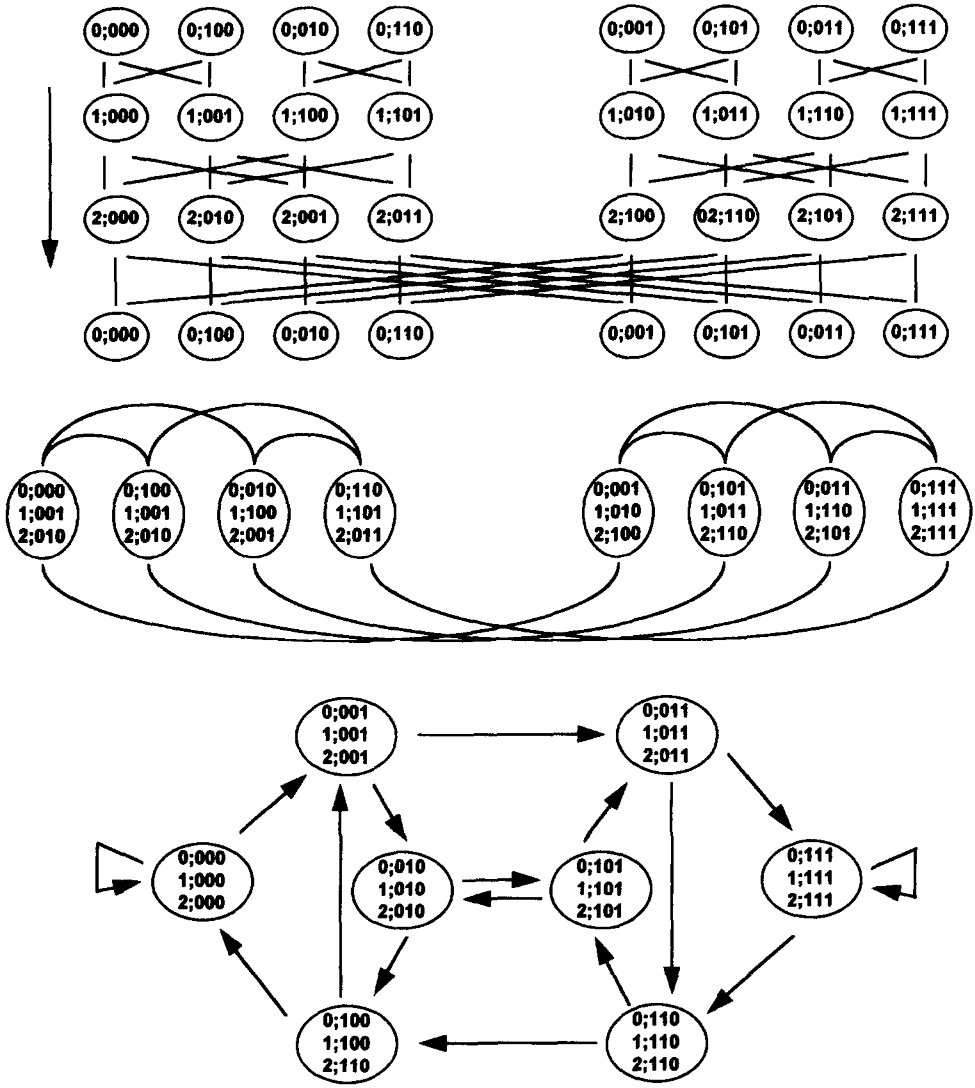

Fig. 2. Algebraic labelings of order-(2, 3) butterfly, hypercube and deBruijn digraphs.

That is, $g_1 g_2 = \langle a + a'; b_{a'} + b'_0, b_{a'+1} + b'_1, \ldots, b_{a'+n-1} + b'_{n-1} \rangle$, where the indices are reduced modulo n.

For each $d > 1$ and $i \in Z_d$, define the element $s_{d,n,i} \in G_{d,n}$ to be

$$s_{d,n,i} = \langle 1; 0, 0, \ldots, i \rangle.$$

The generating set we choose for $G_{d,n}$ is $S_{d,n} = \{s_{d,n,i}: i \in Z_d\}$. In order to simplify the notation, since d and n are always understood from context, we write s_i in place of $s_{d,n,i}$. Let $H_{d,n}$ be the cyclic subgroup of $G_{d,n}$ generated by the element s_0.

Theorem 1 [2]. *The order-(d,n) butterfly digraph is a Cayley digraph. In particular, $\mathscr{B}_{d,n} = \Gamma(G_{d,n}, S_{d,n})$.*

We remark that the labeling induced by the isomorphism of Theorem 1 on the vertices of $\mathscr{B}_{d,n}$ is *different* from the standard one used to define $\mathscr{B}_{d,n}$. This new labeling arises from wreath product multiplication and is crucial in our approach to studying connectivity (see Fig. 2).

Theorem 2 [2]. *The order-(d,n) deBruijn digraph is a right quotient digraph of the order-(d,n) butterfly digraph. In particular, $\mathscr{D}_{d,n} = \Gamma_{\mathrm{R}}(G_{d,n}/H_{d,n}, S_{d,n})$.*

In effect, Theorem 2 allows us to identify the right cosets of $H_{d,n}$ with vertices of $\mathscr{D}_{d,n}$. Each right coset is uniquely defined by a fixed value in the second component. A typical right coset of $H_{d,n}$ in $G_{d,n}$ consists of the n group elements

$$\{\langle a; b_0, b_1, \ldots, b_{n-1}\rangle\colon a \in Z_n\}$$

for some fixed vector $[b_0, b_1, \ldots, b_{n-1}]$.

Finally, we investigate the structure of $\Gamma_{\mathrm{L}}(G_{d,n}/H_{d,n}, S_{d,n})$. A typical left coset of $H_{d,n}$ in $G_{d,n}$ consists of the n group elements

$$\{\langle 0; b_0, b_1, \ldots, b_{n-1}\rangle \cdot s_0^i\colon i \in \{0, \ldots, n-1\}\}$$

for some fixed vector $[b_0, b_1, \ldots, b_{n-1}]$. Each left coset represents a single "column" of the butterfly $\Gamma(G_{d,n}, S_{d,n})$.

Proposition 2. *$\Gamma_{\mathrm{L}}(G_{d,n}/H_{d,n}, S_{d,n})$ is isomorphic to the order-(d,n) hypercube.*

Proof. Let R be the set of all elements of $G_{d,n}$ whose first component is 0. This is a *complete* set of representatives for the left cosets of $H_{d,n}$ in $G_{d,n}$. Now, fix any element $r \in R$, and consider the set of left cosets adjacent in $\Gamma_{\mathrm{L}}(G_{d,n}/H_{d,n}, S_{d,n})$ to the one represented by r. These cosets are of the form

$$r s_0^i s_j H_{d,n},$$

and their representatives in R are of the form

$$r s_0^i s_j s_0^{-(i+1)},$$

where $0 \leqslant i \leqslant n-1$ and $j \in Z_d \setminus \{0\}$. By inspecting this product, we see that r is adjacent to all other representatives that differ in precisely one position of the second component. This is tantamount to the definition of the order-(d,n) hypercube $\mathscr{Q}_{d,n}$. $\square$

Central to our approach for studying connectivity of deBruijn digraphs is the existence of disjoint paths in hypercubes. Sabidussi [12, Lemma 2.3] has proved, via a constructive argument, that $\kappa(\mathscr{G} \times \mathscr{H}) \geqslant \kappa(\mathscr{G}) + \kappa(\mathscr{H})$ for any pair of graphs, $\mathscr{G}$ and $\mathscr{H}$. From this result we deduce the following proposition by induction. (This result was also proved in [11] in the case of the binary hypercube.)

Proposition 3. *The connectivity of $\mathscr{Q}_{d,n}$ equals its degree, i.e., $\kappa(\mathscr{Q}_{d,n}) = n(d-1)$.*

4. Connectivity of right quotient digraphs

We start off with an informal description of our general argument. Suppose $\Gamma(G,S)$ is a Cayley digraph and let H be a subgroup of G. Define $\sigma: G \to G/H$ to be the natural *projection map* that sends each element of G to its right coset of H in G, i.e., for $g \in G$, $\sigma(g) = Hg$. Extend σ to a map between $\Gamma(G,S)$ and $\Gamma_{\mathrm{R}}(G/H,S)$ in the obvious way so that the arc (g, gs) in the Cayley digraph maps to the arc (Hg, Hgs) in the right quotient.

Our aim is to reduce the problem of finding disjoint paths in right quotient digraphs to a similar problem in left digraphs (whose structure is often more readily comprehensible). The key to achieving this rests on the following observation. A path from Hu to Hv in a right quotient digraph avoiding a set, say A, of right cosets, may be constructed by finding a path in the Cayley digraph from u to v avoiding $\sigma^{-1}(A)$, and then projecting this path onto the right quotient digraph. In order to find a path from u to v in the Cayley digraph we appeal to the connectivity of the *left quotient* and attempt to avoid *all* left cosets that contain some vertex of $\sigma^{-1}(A)$. By Proposition 1, assuming the induced subgraph on H is connected, such a path in $\Gamma_{\mathrm{L}}(G/H,S)$ can be converted to an appropriate path in $\Gamma(G,S)$ by splicing in a single path (completely contained in some left coset) for each vertex in the path in $\Gamma_{\mathrm{L}}(G/H,S)$.

We now implement the above idea in the case of deBruijn digraphs.

Theorem 3 [5, 15, 4]. *The connectivity of the deBruijn digraphs is equal to their minimum degree, i.e., $\kappa(\mathscr{D}_{d,n}) = d-1$.*

Proof. For notational convenience, we fix d and n, and put $G = G_{d,n}$, $H = H_{d,n}$, and $S = S_{d,n}$. Let Hu and Hv be any pair of distinct right cosets of H (we are regarding them as vertices of $\Gamma_{\mathrm{R}}(G/H,S)$). Choose a set, say A, of $d-2$ other distinct right cosets. We aim to find a path in $\Gamma_{\mathrm{R}}(G/H,S)$ from Hu to Hv avoiding A.

Define a *weight* function, $w: G \to N$, that maps group elements to natural numbers, as follows. If $g = \langle a; b_0, b_1, \ldots, b_{n-1}\rangle \in G$ then

$$w(g) = \sum_{i=0}^{n-1} b_i,$$

where the b_i are treated as natural numbers (with the usual addition) while taking the sum. Note that the weight of all elements in a particular right coset of H is a constant (by definition of H). Furthermore, the weight of any element in a particular left coset of H is constant since one element of a left coset is obtained from another by rotating the vector $[b_0, b_1, \ldots, b_{n-1}]$ (together with an appropriate change in a), leaving the weight unchanged. Thus, we extend the definition of weight to left cosets and right cosets by defining

$$w(Hg) \stackrel{\text{def}}{=} w(gH) \stackrel{\text{def}}{=} w(g).$$

The following two lemmas describe an important property of weights.

Lemma 1. *Let Hg be an arbitrary right coset. Then, the weights of the right cosets Hgs_i for $i \in Z_d$ are distinct. Similarly, the weights of the left cosets gs_iH for $i \in Z_d$ are distinct.*

Proof. Let $g = \langle a; b_0, b_1, \ldots, b_{n-1} \rangle$. Then,

$$gs_i = \langle a + 1; b_1, b_2, \ldots, b_{n-1}, b_0 + i \rangle,$$

and if $i \neq j$ then $b_0 + i \neq b_0 + j$, implying that $w(gs_i) \neq w(gs_j)$ and hence $w(Hgs_i) \neq w(Hgs_j)$. Since $w(Hx) = w(xH)$ for any element $x \in G$, the remainder of the lemma follows. $\square$

The proof of the next lemma is analogous to the proof of Lemma 1.

Lemma 2. *Let Hg be an arbitrary right coset. Then, the weights of the right cosets Hgs_i^{-1} for $i \in Z_d$ are distinct. Similarly, the weights of the left cosets $gs_i^{-1}H$ for $i \in Z_d$ are distinct.*

Returning to the proof of the theorem, let $A' = \sigma^{-1}(A)$ (recall that σ denotes the natural projection map from G to the set of right cosets). Notice that $|A'| = n(d-2)$ since $|H| = n$ and $|A| = d - 2$. Now consider two cases.

Case 1: *Assume* $A' \cap (uH \cup vH) = \emptyset$. By Proposition 3, $\kappa(\Gamma_L(G/H, S)) = n(d-1)$, and hence we can find a path from uH to vH avoiding any $n(d-1) - 1$ left cosets of H *entirely*. But A' can intersect at most $n(d-2)$ left cosets since its cardinality is $n(d-2)$. Hence, since $n(d-1) - 1 \geqslant n(d-2)$ for all $n \geqslant 1$, it is transparent that we can avoid all vertices in A', proving the theorem in this case.

Case 2: *Assume* $A' \cap (uH \cup vH) \neq \emptyset$. Suppose, in particular, that $A' \cap uH \neq \emptyset$. We reduce this case to the previous by showing there is a left coset, usH for some $s \in S \backslash \{s_0\}$, which does not intersect A'. First, observe that the elements of A' assume at most $d - 2$ distinct weights since all elements in a fixed right coset have equal weight. Hence, using Lemma 1 and the pigeon-hole principle, they cannot belong to more than $d - 2$ of the left cosets usH ($s \in S$). Thus, there exists $s \in S$ such that $usH \cap A' = \emptyset$. By a similar argument, we find $t \in S$ such that $vt^{-1}H \cap A' = \emptyset$ (using Lemma 2). From Case 1 we can find a path from usH to $vt^{-1}H$ avoiding all left cosets involving A'. The projection of this path into the right quotient is a path from Hus to Hvt^{-1} avoiding A. Since (Hu, Hus) and (Hvt^{-1}, Hv) are both arcs in the right quotient, the above path can be extended to a path from Hu to Hv avoiding A, as desired. □

5. Extension I – avoiding shuffle cycles

On closer inspection, the proof of Theorem 3 actually yields a stronger result. Define a *shuffle cycle* to be any simple cycle in $\mathscr{D}_{d,n}$ consisting only of shuffle arcs. Thus, a typical shuffle cycle traverses the following path:

$$[b_0, b_1, \ldots, b_{n-1}] \to [b_1, \ldots, b_{n-1}, b_0] \to \cdots \to [b_0, b_1, \ldots, b_{n-1}].$$

A shuffle cycle is said to have *length* k if it involves exactly k vertices. Let $\mathscr{C}$ denote the set of vertices in any shuffle cycle in $\mathscr{D}_{d,n}$ of length k. Then $\sigma^{-1}(\mathscr{C})$ is a collection of group elements which lie in k distinct left cosets of H. This follows from the fact that any left coset intersecting $\sigma^{-1}(\mathscr{C})$ contains exactly n/k representatives of *every* right coset in the shuffle cycle. This observation leads to the following result.

Theorem 4. *Let Hu, Hv be any 2 distinct vertices of $\mathscr{D}_{d,n}$. Let $\mathscr{F}$ be the set of vertices comprising a family of shuffle cycles of total length less than $n(d-1)$ such that neither Hu nor Hv lies on any of these cycles. Then, there is a path from Hu to Hv that avoids $\mathscr{F}$.*

Proof. By the observation preceding the theorem and the assumption of the theorem, we have that $\sigma^{-1}(\mathscr{F})$ lies in fewer than $n(d-1)$ left cosets of H. Also, since *every* right coset in a shuffle cycle is represented by at least one element in every left coset involved in $\sigma^{-1}(\mathscr{F})$, if Hu and Hv do not lie on any of the cycles in $\mathscr{F}$ then uH and vH do not intersect $\sigma^{-1}(\mathscr{F})$. By appealing to the connectivity of $\Gamma_L(G/H, S)$, we find a path from uH to vH avoiding $\sigma^{-1}(\mathscr{F})$. The projection of this path onto $\mathscr{D}_{d,n}$ is a path from Hu to Hv avoiding $\mathscr{F}$. □

Remark. This result is best possible, since removing the length-n shuffle cycles involving the $d-1$ vertices of the form $[0, 0, \ldots, i]$, $1 \leqslant i \leqslant d-1$, will obviously disconnect $\mathscr{D}_{d,n}$, and their total length equals $n(d-1)$.

5.1. An example

Notice that Theorems 3 and 4 are *constructive* in terms of an algorithm for avoiding a set of vertices in $\mathscr{D}_{d,n}$. In effect, we have reduced the connectivity problem in deBruijn digraphs to the connectivity problem in hypercube digraphs. Consider $\mathscr{D}_{3,4}$ and suppose $u = [2, 2, 1, 0]$, $v = [2, 0, 0, 1]$ and let $A = \{[0, 0, 1, 1], [2, 0, 2, 0]\}$. Then,

$$\sigma^{-1}(A) = \{\langle i; 0, 0, 1, 1\rangle : i \in Z_2\} \cup \{\langle i; 2, 0, 2, 0\rangle : i \in Z_2\}.$$

The set of left cosets involved in $\sigma^{-1}(A)$ is

$$A'' = \{[0,0,1,1], [0,1,1,0], [1,1,0,0], [1,0,0,1], [2,0,2,0], [0,2,0,2]).$$

We proceed by finding a path, denoted $\boldsymbol{P}$, in $\mathcal{Q}_{3,4}$ from $[2,2,1,0]$ to $[2,0,0,1]$ avoiding A'':

$$\boldsymbol{P} := \mathbf{[2,2,1,0]} \to [2,2,1,1] \to [2,2,0,1] \to \mathbf{[2,0,0,1]}.$$

Finally, we project $\boldsymbol{P}$ to a path in $\mathscr{D}_{3,4}$, splicing in a path for each left coset traversed by $\boldsymbol{P}$:

$\mathbf{[2,2,1,0]}$	$\to [2,2,1,1]$	$\to [1,2,2,0]$	$\to [0,1,2,0]$
$\to [2,1,0,2]$	$\to [2,1,1,2]$	$\to [2,2,0,1]$	$\to [1,2,0,0]$
$\to [1,0,2,2]$	$\to [1,1,2,2]$	$\to [2,0,1,2]$	$\to \mathbf{[2,0,0,1]}$
$\to [0,2,2,1]$			

6. Extension II – (Hu, Hv)-connectivity

Every vertex of the d-ary deBruijn digraph has degree d except for vertices of the form $[b, b, \ldots, b]$, for some $b \in \{0, 1, \ldots, d-1\}$. These vertices have a self-loop and are of degree $d-1$. Now suppose that Hu and Hv are 2 distinct vertices of $\mathscr{D}_{d,n}$, both of degree d. It is natural to ask whether, in this specific case, the (Hu, Hv)-connectivity equals d. The next theorem proves that this is true except in a special case. To simplify the statement, we make the following definition. Let Hx be a vertex of $\mathscr{D}_{d,n}$.

- If Hx is adjacent to a vertex of degree $d-1$, then Hx is said to be *out-special.*
- If Hx is adjacent from a vertex of degree $d-1$, then Hx is said to be *in-special.*

Theorem 5. *Let Hu and Hv be two distinct vertices of $\mathscr{D}_{d,n}$ of degree d. Then, $\kappa(Hu, Hv) = d$ if and only if Hu is not out-special and Hv is not in-special.*

Proof. A digraph whose connectivity equals its degree is called *superconnected* if the only way the digraph can minimally lose the property of strong connectivity is to remove all vertices incident from (or to) a single vertex. Results of Hamidoune et al. [8] provide a characterization of superconnected abelian Cayley digraphs. It follows directly from this characterization that the quotient digraph $\Gamma_{\mathrm{L}}(G/H, S)$ is superconnected since it is isomorphic to the direct product of complete symmetric digraphs. Now choose 2 distinct right cosets of H, say Hu and Hv, each of degree d, and let A be a set of any other $d-1$ right cosets of H. We distinguish two cases.

Case 1: *Assume* $(uU \cup vH) \cap \sigma^{-1}(A) = \emptyset$. Take $A' = \sigma^{-1}(A)$. Following the approach of Theorem 3 a path from uH to vH in $\Gamma_{\mathrm{L}}(G/H, S)$ avoiding A' does not necessarily exist since $|A'| = n(d-1)$ and $\kappa(\Gamma_L(G/H,S)) = n(d-1)$. But, by the superconnectivity of $\Gamma_L(G/H,S)$, in order to disconnect $\Gamma_L(G/H,S)$ and thus foil our attempt to find a path from uH to vH the set of left cosets comprising A' must all be adjacent to either uH or vH. We proceed by investigating when this arrangement can occur. Assume the left cosets comprising A' are all adjacent to uH. (The case in which the set of left cosets comprising A' are all adjacent from vH is analogous.)

First, recall that $uH = \{us_0^i\colon 0 \leqslant i \leqslant n-1\}$. The set of left cosets adjacent to uH is

$$\mathscr{N} \stackrel{\text{def}}{=} \{us_0^i s_j H\colon 0 \leqslant i \leqslant n-1, j \in Z_d \setminus \{0\}\}.$$

Since A' consists of $d-1$ weights and, by assumption, there is an element of A' in *every* left coset of $\mathscr{N}$, there can be at most $d-1$ weights involved in $\mathscr{N}$. We now show that this can occur only if the second component of u is of the form $[b, b, \ldots, b]$ for some $b \in Z_d$.

Denote the second component of u by $[a, b, X]$ where $a, b \in Z_d$ and X is an $(n-2)$-tuple of elements in Z_d. Consider the set of vertices adjacent to u but not in uH. Their second components have the form $[b, X, a']$, where $a' \in Z_d$, $a' \neq a$, and thus each of these elements has weight equal to

$$b + a' + w(X).$$

Denote this set of weights by $\mathscr{W}_1$.

Now consider the vertex us_0. It is the unique vertex adjacent to u in uH. The second component of us_0 is $[b, X, a]$. Consider, as before, the set of vertices adjacent to us_0 but not in uH. Their second components have the form $[X, a, b']$, where $b' \in Z_d$ and $b' \neq b$. The weights of these elements equals

$$a + b' + w(X).$$

Denote this set of weights by $\mathscr{W}_2$. Because there are only $d-1$ right cosets involved in A', we must have $\mathscr{W}_1 = \mathscr{W}_2$. Now suppose that $a \neq b$, and furthermore, assume without loss of generality, that $a < b$. Choose $a' = d-1$. (This does not contradict the fact that $a' \neq a$ since, by assumption, $a < b$.) Then, for any $b' \in Z_d$,

$$a + b' < b + b' \leqslant b + a',$$

where the last inequality follows from the observation that $b' \leqslant a'$ since $a' = d-1$. Hence, there is a weight in $\mathscr{W}_1$ which is larger than any weight in $\mathscr{W}_2$, contradicting the assertion that $\mathscr{W}_1 = \mathscr{W}_2$. This proves that $a = b$.

We proceed by applying this argument repeatedly to each pair of elements us_0^i, us_0^{i+1} $(i = 1, 2, \ldots, n-1)$, ultimately showing that the second component of u is of the form $[b, b, \dots, b]$, for some $b \in Z_d$. It follows that $Hus_0 = Hu$, proving that Hu is of degree $d-1$.

In summary, we have shown that in order to have exactly $d-1$ right cosets intersecting *all* the left cosets neighboring some fixed left coset uH, the right coset Hu must be of degree $d-1$. Therefore, as long as Hu and Hv are of degree d, we can find a path in $\Gamma(G, S)$ from u to v avoiding A'. The projection of this path onto $\Gamma_{\mathrm{R}}(G/H, S)$ is a path from Hu to Hv avoiding A.

Case 2: *Assume* $(uH \cup vH) \cap \sigma^{-1}(A) \neq \emptyset$. Suppose that $uH \cap \sigma^{-1}(A) \neq \emptyset$. (The case in which $vH \cap \sigma^{-1}(A) \neq \emptyset$ is analogous.) In this case, by Lemma 1, we infer the existence of an element adjacent to u in $\Gamma(G, S)$, say us $(s \in S \setminus \{s_0\})$, which is in a left coset that does not intersect A'. Hence, as long as Hus is of degree d, the previous case applies. The reverse implication of the theorem now follows immediately from Cases 1 and 2.

The forward implication is verified via the following observation: Suppose Hu is out-special. Without loss of generality, let this vertex be $Hu = [1, 0, \dots, 0]$. By direct calculation all paths of length at least 2 originating at Hu will pass through the set of $d-1$ vertices $\{[0, \dots, 0, i]\colon 1 \leqslant i \leqslant d-1\}$ and hence there can be at most $d-1$ disjoint paths from Hu to $[1, 1, \dots, 1]$. (In effect, this set of vertices is a cutset for $\mathscr{D}_{d,n}$.) If Hv is in-special, an analogous argument applies. $\square$

7. Connectivity of shuffle-exchange digraphs

In this section, we discuss the connectivity of shuffle-exchange digraphs. The techniques developed in previous sections for the deBruijn digraphs can be directly instantiated (after appropriately defining the generating set for $G_{d,n}$) yielding similar results. We now describe the algebraic structure of the shuffle-exchange digraphs and continue by a brief discussion of the results.

Shuffle-exchange digraphs: The order-(d, n) *shuffle-exchange digraph* $(d \geqslant 2,\ n \geqslant 1)$, denoted $\mathscr{S}\mathscr{E}_{d,n}$, has a vertex set consisting of all n-tuples, $[b_0, b_1, \dots, b_{n-1}]$, where each $b_i \in \{0, 1, \dots, d-1\}$. There are 2 types of arcs. The *shuffle arcs* connect $[b_0, b_1, \dots, b_{n-1}]$ to $[b_1, \dots, b_{n-1}, b_0]$. The *exchange arcs* connect $[b_0, b_1, \dots, b_{n-1}]$ to $[b_0, b_1, \dots, b']$, where $b' \neq b_{n-1}$.

Cube-connected cycles digraphs: The order-(d, n) *cube-connected cycles digraph* $(d \geqslant 2, n \geqslant 1)$, denoted $\mathscr{B}_{d,n}$, has a vertex set consists of the set of all pairs $\langle l; B \rangle$, where $l \in \{0, \dots, n-1\}$ and B is an n-tuple $[b_0, b_1, \dots, b_{n-1}]$ with each $b_i \in \{0, 1, \dots, d-1\}$. There is an arc from $\langle l_1; B_1 \rangle$ to $\langle l_2; B_2 \rangle$ if $l_2 = l_1 + 1 \bmod n$ and $B_1 = B_2$ or $l_2 = l_1$ and $B_1 = [b_0, \dots, b_{l_1}, \dots, b_{n-1}]$, and $B_2 = [b_0, \dots, b', \dots, b_{n-1}]$, for any $b' \in \{0, 1, \dots, d-1\}$.

Define

$$t_{d,n,i} = \begin{cases} \langle 1; 0, 0, \dots, 0 \rangle & \text{if } i = 0, \\ \langle 0; 0, \dots, 0, i \rangle & \text{if } i \in Z_d \setminus \{0\}. \end{cases}$$

We need the following facts describing the algebraic structure of the shuffle-exchange digraphs:

1. If we choose $T_{d,n} = \{t_{d,n,i}\colon i \in Z_d\}$, then $\mathscr{C}_{d,n}$ is isomorphic to $\Gamma(G_{d,n}, T_{d,n})$.
2. Let $H_{d,n}$ denote the subgroup generated by $t_{d,n,0}$. Then the right quotient digraph, $\Gamma_{\mathrm{R}}(G_{d,n}/H_{d,n}, T_{d,n})$, is isomorphic to the order-(d,n) shuffle-exchange digraph.
3. The left quotient digraph, $\Gamma_{\mathrm{L}}(G_{d,n}/H_{d,n}, T_{d,n})$, is isomorphic to the order-(d,n) hypercube.

The first two statements were proved in [2]. The proof of Statement 3 is analogous to that of Proposition 2. Using these facts, we apply the same arguments used in Theorems 3–5 to obtain the next three results.

Theorem 6. *The connectivity of the shuffle-exchange digraphs is equal to their minimum degree, i.e.,* $\kappa(\mathscr{S}\mathscr{E}_{d,n}) = d - 1$.

Theorem 7. *Let* Hu, Hv *be any* 2 *distinct vertices of* $\mathscr{S}\mathscr{E}_{d,n}$. *Let* $\mathscr{F}$ *be the set of vertices comprising a family of shuffle cycles of total length* $< n(d-1)$ *such that neither* Hu *nor* Hv *lies on any of these cycles. Then, there is a path from* Hu *to* Hv *that avoids* $\mathscr{F}$.

Since the exchange arcs always form a pair of mutually opposing arcs, we need a slightly different statement for the analogue to Theorem 5. Define a vertex to be *special* if it is adjacent to a vertex of degree $d - 1$.

Theorem 8. *Let* Hu *and* Hv *be two distinct vertices of* $\mathscr{S}\mathscr{E}_{d,n}$ *of degree* d. *Then,* $\kappa(Hu, Hv) = d$ *if and only if both* Hu *and* Hv *are not special.*

8. Future extensions

There are two main areas in which we foresee extensions to our results. First, we would like to adapt the ideas in this paper in order to handle *undirected* deBruijn graphs. Second, we are interested in deriving a generalized version of the main technique developed for studying connectivity (cf. Theorem 3). This would allow us to determine connectivity results for classes of regular digraphs (i.e., where the in-degree equals the out-degree) based on the groups with which they are associated. (It has been shown that *all* regular digraphs can be expressed as right quotient digraphs of Cayley digraphs [2, 6].) The most difficult part of this research is identifying appropriate graph-theoretic conditions that give rise to specific group-theoretic structure.

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